FAR BEYOND

MAT122

Fundamental Theorem of Calculus (FTC)

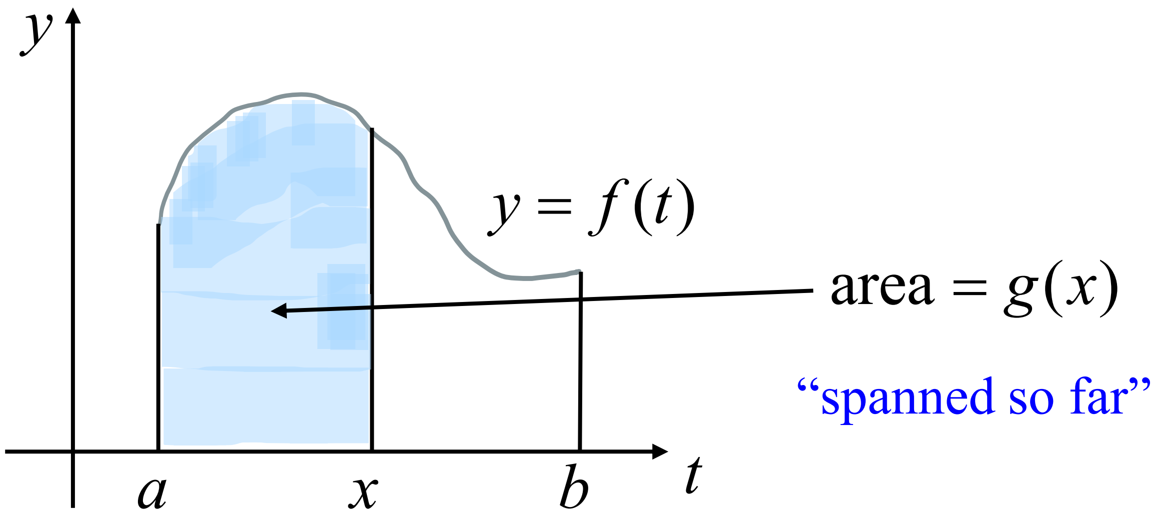
Stony Brook University

Fundamental Theorem of Calculus

The **Fundamental Theorem of Calculus** establishes a connection between differential calculus and integral calculus. They are inverse processes.

FTC Part 1: $\int_a^x f(t)dt = g(x)$ where f is continuous on $[a, b]$.

If f is a positive function then $g(x)$ can be interpreted as the area under the graph of f from a to x where x can vary from a to b.

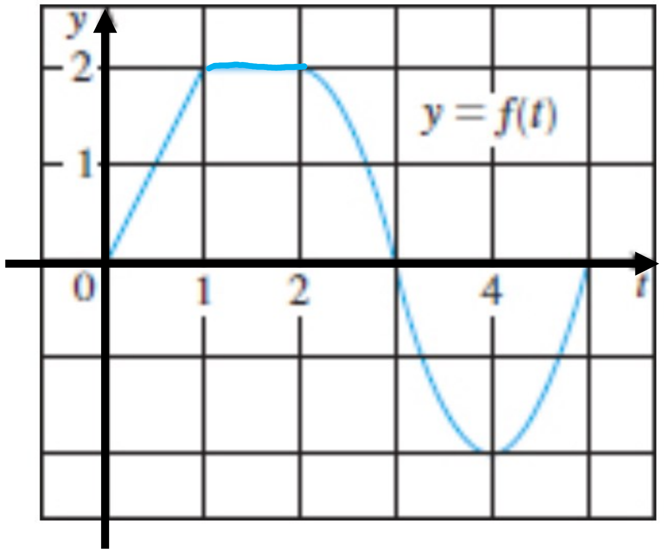


i.e., $g(x)$ is an antiderivative of f :

$$g'(x) = f(x)$$

for $a < x < b$

FTC Part 1 - Example



graph is represented by $g(x) = \int_0^x f(t)dt$

Find value of $g(0) = \int_0^0 f(t)dt = \boxed{0}$

Find value of $g(1) = \int_0^1 f(t)dt = A_{\triangle} = \frac{1}{2}(1)(2) = \boxed{1}$

Find value of $g(2) = \int_0^2 f(t)dt = A_{\triangle} + A_{\square} = 1 + (1)(2) = \boxed{3}$

Find value of $g(3) = \int_0^3 f(t)dt = \int_0^2 f(t)dt + \int_2^3 f(t)dt$

$= 3 + \sim 1.3 = \boxed{4.3}$

Find value of $g(4) = \int_0^3 f(t)dt + \int_3^4 f(t)dt$

$= 4.3 + -1.3 = \boxed{3}$

Find value of $g(5) = g(4) + \int_4^5 f(t)dt$

$= 3 + -1.3 = \boxed{1.7}$

Differentiating an Integral

FTC Part 1:

$$\int_a^x f(t)dt = g(x)$$

where f is continuous

ex. Find the derivative of $g(x) = \int_0^x \sqrt{1+t^2}\ dt$

since $f(t)$ is continuous on $x \geq 0$

then, by FTC Part 1,

$$g'(x) = \sqrt{1+x^2}$$

ex. Find $S'(x)$ when $S(x) = \int_0^x 7t^3\ dt$

$7t^3$ is continuous everywhere

$$\therefore S'(x) = 7x^3$$

Differentiating an Integral with Chain Rule

When upper bound is not a simple variable "x", chain rule is necessary.

ex. Find $\dfrac{d}{dx}\int_1^{x^4} \ln t \, dt$

note: can be **any** constant

let $x^4 = u$: $\dfrac{d}{dx}\int_1^{u} \ln t \, dt$

$$\frac{d}{du}\int_1^{u} \ln t \, dt \cdot \frac{du}{dx}$$

$$\underbrace{\frac{d}{du}\left(\int_1^{u} \ln t \, dt\right)}\cdot \frac{du}{dx}$$

plug in u

$$\ln u \cdot \frac{du}{dx}$$

plug back in for x

$$\ln x^4 \cdot 4x^3 = \boxed{4x^3 \ln x^4}$$

Chain Rule without u-substitution

When upper bound is not a simple variable "x", chain rule is necessary.

ex. Find $\frac{d}{dx}\int_7^{x^2}\left(3+t^2\right)\,dt$

ignore

$$=\left[\int_7^{x^2}\left(3+t^2\right)\,dt\right]'\cdot\left(x^2\right)'$$

$$=\left(3+\left(x^2\right)^2\right)\cdot 2x$$

simplify

$$=\boxed{2x\left(3+x^4\right)}$$

FTC with Negation

ex. Find the derivative of $\int_x^{\pi} \sqrt{1+e^t}\ dt$

constant is upper bound

$$= -\int_{\pi}^{x} \sqrt{1+e^t}\ dt$$

$$= \boxed{-\sqrt{1+e^x}}$$

negate

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

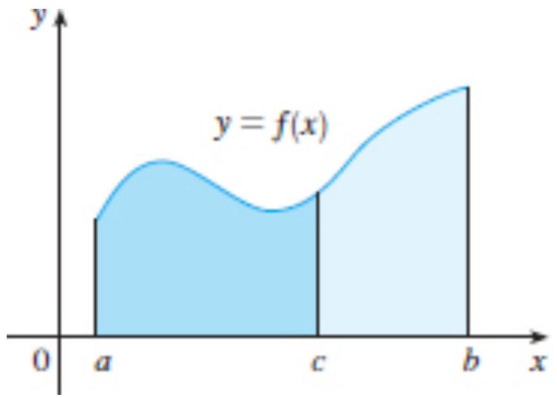
switch *a* and *b*

FTC with Variables in Both Bounds

both bounds are variables

ex. Find the derivative of $\int_{2x}^{3x} \frac{u^2-1}{u^2+1}\,du$ break into two integrals:

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$



$$= \int_{2x}^{0} \frac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\,du$$

$$= -\int_{0}^{2x} \frac{u^2-1}{u^2+1}\,du + \int_{0}^{3x} \frac{u^2-1}{u^2+1}\,du$$

plug in $2x$, $3x$

$$= -\frac{(2x)^2-1}{(2x)^2+1}\cdot 2 + \frac{(3x)^2-1}{(3x)^2+1}\cdot 3$$

distribute squares

chain rule

$$= \boxed{\frac{3(9x^2-1)}{9x^2+1} - \frac{2(4x^2-1)}{4x^2+1}}$$

FTC - Do

Do: Find the derivative of $\int_{x^3}^{1} \sqrt{7t^2 - 3t + 6}\, dt$

Do: Find the derivative of $\int_{4x}^{9x} \ln t\, dt$